Mechanics of Solids: Anisotropy

The Hook’s law for the general material is presented

According to the material’s structural symmetries,
we deduce the number of elastic constants in various types of anisotropy

Supporting material
Appendix D: Notes on Anisotropy (J Botsis)

Mechanics of Solids: Anisotropy

A material is *linearly elastic* if its applied stress field → $\sigma(\boldsymbol{x})$

is related to the resulting strain field → $\varepsilon(\boldsymbol{x})$

by a linear relation or the generalized Hooke law,

$$\sigma(\boldsymbol{x}) = \boldsymbol{C}(\boldsymbol{x})\varepsilon(\boldsymbol{x}) \quad , \quad \sigma_{kl}(\boldsymbol{x}) = C_{klmn}(\boldsymbol{x})\varepsilon_{mn}(\boldsymbol{x}) \qquad (k,l,m,n=1,2,3)$$

Stiffness fourth order tensor

$$C'_{klmn}(\boldsymbol{x}) = c_{ki}c_{lj}c_{mp}c_{nq}C_{ijpq}(\boldsymbol{x}) \qquad (k,l,m,n,i,j,p,q=1,2,3)$$

Mechanics of Solids: Anisotropy

$$\sigma(\boldsymbol{x}) = \boldsymbol{C}(\boldsymbol{x})\varepsilon(\boldsymbol{x}) \quad , \quad \sigma_{kl}(\boldsymbol{x}) = C_{klmn}(\boldsymbol{x})\varepsilon_{mn}(\boldsymbol{x}) \qquad (k,l,m,n=1,2,3)$$

$$C'_{klmn}(\boldsymbol{x}) = c_{ki}c_{lj}c_{mp}c_{nq}C_{ijpq}(\boldsymbol{x}) \qquad (k,l,m,n,i,j,p,q=1,2,3)$$

When the stiffness $\boldsymbol{C}(\boldsymbol{x})$ is independent of $\boldsymbol{x}$ the we have a homogeneous material otherwise it is inhomogeneous.

In the general case we have 9 equations and $3^4 = 81$ elastic constants.

$$\sigma_{11} = C_{1111}\varepsilon_{11} + C_{1112}\varepsilon_{12} + C_{1113}\varepsilon_{13} + C_{1121}\varepsilon_{21} + C_{1122}\varepsilon_{22} + C_{1123}\varepsilon_{23} + C_{1131}\varepsilon_{31} + C_{1132}\varepsilon_{32} + C_{1133}\varepsilon_{33}$$

$$\sigma_{22} = \ldots\ldots.$$

Mechanics of Solids: Anisotropy

$$\sigma_{11} = C_{1111}\varepsilon_{11} + C_{1112}\varepsilon_{12} + C_{1113}\varepsilon_{13} + C_{1121}\varepsilon_{21} + C_{1122}\varepsilon_{22} + C_{1123}\varepsilon_{23} + C_{1131}\varepsilon_{31} + C_{1132}\varepsilon_{32} + C_{1133}\varepsilon_{33}$$

$$\sigma_{22} = \ldots\ldots\ldots$$

In matrix form these equation are

Stiffness Matrix

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{pmatrix} = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1113} & C_{1123} \\ C_{2211} & C_{2222} & C_{2233} & C_{2212} & C_{2213} & C_{2223} \\ C_{3311} & C_{3322} & C_{3333} & C_{3312} & C_{3313} & C_{3323} \\ C_{1211} & C_{1222} & C_{1233} & C_{1212} & C_{1213} & C_{1223} \\ C_{1311} & C_{1322} & C_{1333} & C_{1312} & C_{1313} & C_{1323} \\ C_{2311} & C_{2322} & C_{2333} & C_{2312} & C_{2313} & C_{2323} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{pmatrix}$$

We can write these equations using compliance coefficients by taking the inverse of the stiffness

$$\boldsymbol{\varepsilon} = \boldsymbol{S\sigma} \quad , \quad \varepsilon_{kl} = S_{klmn}\sigma_{mn}$$

$$\boldsymbol{S} = \boldsymbol{C}^{-1}$$

Compliance Matrix

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{pmatrix} = \begin{pmatrix} S_{1111} & S_{1122} & S_{1133} & S_{1112} & S_{1113} & S_{1123} \\ S_{2211} & S_{2222} & S_{2233} & S_{2212} & S_{2213} & S_{2223} \\ S_{3311} & S_{3322} & S_{3333} & S_{3312} & S_{3313} & S_{3323} \\ S_{1211} & S_{1222} & S_{1233} & S_{1212} & S_{1213} & S_{1223} \\ S_{1311} & S_{1322} & S_{1333} & S_{1312} & S_{1313} & S_{1323} \\ S_{2311} & S_{2322} & S_{2333} & S_{2312} & S_{2313} & S_{2323} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ 2\sigma_{12} \\ 2\sigma_{13} \\ 2\sigma_{23} \end{pmatrix}$$

Mechanics of Solids: Anisotropy

SYMMETRIES OF THE STIFFNESS MATRIX (OR COMPLIANCE)

Due to the symmetries of the stress and strain tensors, the

$$\sigma_{kl} = \sigma_{lk}$$
$$\varepsilon_{mn} = \varepsilon_{nm}$$

$\Rightarrow$ $C_{klmn} = C_{lkmn} = C_{klnm}$ are called minor symmetries

These symmetries reduce the number of independent constants:

From $3^4 = 81$ to 36.

To proceed further we need the help of thermodynamics:

Mechanics of Solids: Anisotropy

SYMMETRIES OF THE STIFFNESS MATRIX (OR COMPLIANCE)

In linear elasticity we adopt two important hypotheses:

1: For an adiabatic or isothermal process,
there exists a *strain energy density function W,* which is also a potential for the stresses.

strain energy density

$$\sigma_{ij} = \frac{\partial W(\varepsilon_{ij})}{\partial \varepsilon_{ij}}$$

stress energy density

$$\left(\varepsilon_{ij} = \frac{\partial W^*(\sigma_{ij})}{\partial \sigma_{ij}} \right)$$

2: The stability hypothesis which states that the stiffness tensor is *positive definite*, i.e.,

$$\varepsilon : \boldsymbol{C}\varepsilon = C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} > 0, \quad \forall \quad \varepsilon_{ij} \neq 0 \Rightarrow W(\boldsymbol{\varepsilon}_{kl}) = \frac{1}{2} C_{ijkl}\boldsymbol{\varepsilon}_{ij}\boldsymbol{\varepsilon}_{kl}$$

$$\left(W^*(\sigma_{kl}) = \frac{1}{2} S_{ijkl}\sigma_{ij}\sigma_{kl} \right)$$

These two hypotheses result in:

$$\frac{\partial W}{\partial \varepsilon_{11}} = \sigma_{11} = C_{1111}\varepsilon_{11} + C_{1122}\varepsilon_{22} + ... + C_{1132}\varepsilon_{32}$$

$$\frac{\partial W}{\partial \varepsilon_{22}} = \sigma_{22} = C_{2211}\varepsilon_{11} + C_{2222}\varepsilon_{22} + ... + C_{2232}\varepsilon_{32}$$

$$\frac{\partial^2 W}{\partial \varepsilon_{11}\partial \varepsilon_{22}} = C_{1122} = C_{2211} \Rightarrow \frac{\partial^2 W}{\partial \varepsilon_{kl}\partial \varepsilon_{mn}} = C_{klmn} = C_{mnkl}$$

This symmetry reduces the independent elastic constants to 21
(the same arguments apply for the compliance coefficients)

SYMMETRIES OF THE STIFFNESS MATRIX (OR COMPLIANCE)

Overall the symmetries are: $C_{klmn} = C_{lkmn} = C_{klnm} = C_{mnkl}$

With these symmetries, the matrix elements can be simplified as shown in the table below,

tensor notation	11	22	33	23,32	31,13	12,21
matrix notation⁺	1	2	3	4	5	6

⁺This notation is used in the literature for layered composite materials

Mechanics of Solids: Anisotropy

BASIC CASES OF ELASTIC SYMMETRY

With respect to their elastic properties, all engineering materials can be divided into *isotropic* and *anisotropic*.

The symmetry of an elastic material depends upon the symmetry of its structure.

The relationship between the structural and elastic symmetry for crystals was established according to F. Neumann's principle:

"*the symmetry of the elastic properties of a solid contains that of its crystallographic structure*".

The elastic symmetry in an anisotropic material, renders Hook's law simpler since some of the coefficients are related or are zero

Mechanics of Solids: Anisotropy

BASIC CASES OF ELASTIC SYMMETRY

How we proceed:

We express, stress-strain relations (or the stiffness matrix) with respect to $Ox_1x_2x_3$ $\quad \sigma_{kl} = C_{klmn}\varepsilon_{mn}$

We express the same relations with respect to the system $Ox_1^{'}x_2^{'}x_3^{'}$ $\quad \sigma'_{kl} = C'_{klmn}\varepsilon'_{mn}$

These two systems possess the symmetries of the investigated structure

We transform the stiffness tensor one on to another.

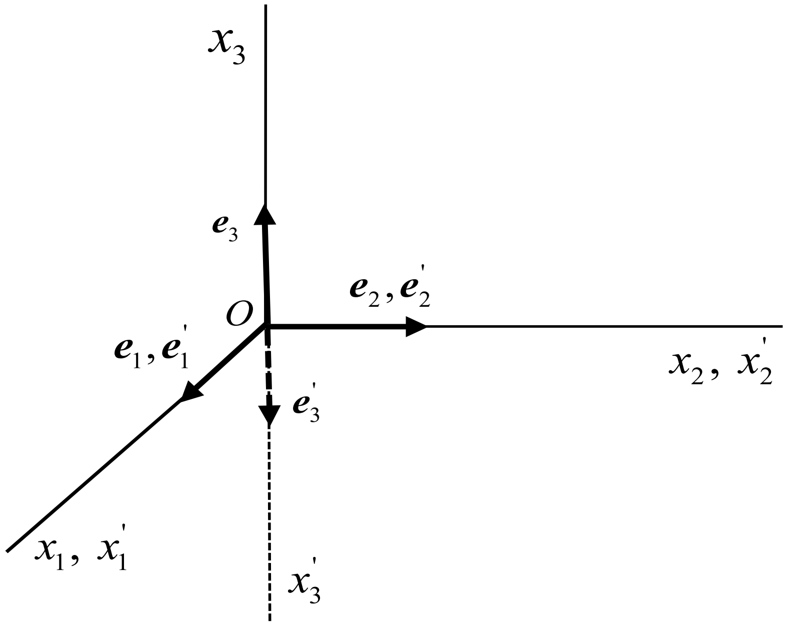
$$C_{klmn}^{'} = c_{ki}c_{lj}c_{mp}c_{nq}C_{ijpq}$$

We compare and deduce the constants.
(the same procedure applies for the compliance $\varepsilon_{kl} = S_{klmn}\sigma_{mn}$)

Mechanics of Solids: Anisotropy

BASIC CASES OF ELASTIC SYMMETRY

1: symmetry with respect to one plane: *material is defined as monoclinic*



$$\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{13}\\ \sigma_{23}\end{pmatrix} = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & 0 & 0\\ C_{1122} & C_{2222} & C_{2233} & C_{2212} & 0 & 0\\ C_{1133} & C_{2233} & C_{3333} & C_{3312} & 0 & 0\\ C_{1112} & C_{2212} & C_{3312} & C_{1212} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{1313} & C_{1323}\\ 0 & 0 & 0 & 0 & C_{1323} & C_{2323}\end{pmatrix}\begin{pmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{12}\\ 2\varepsilon_{13}\\ 2\varepsilon_{23}\end{pmatrix}$$

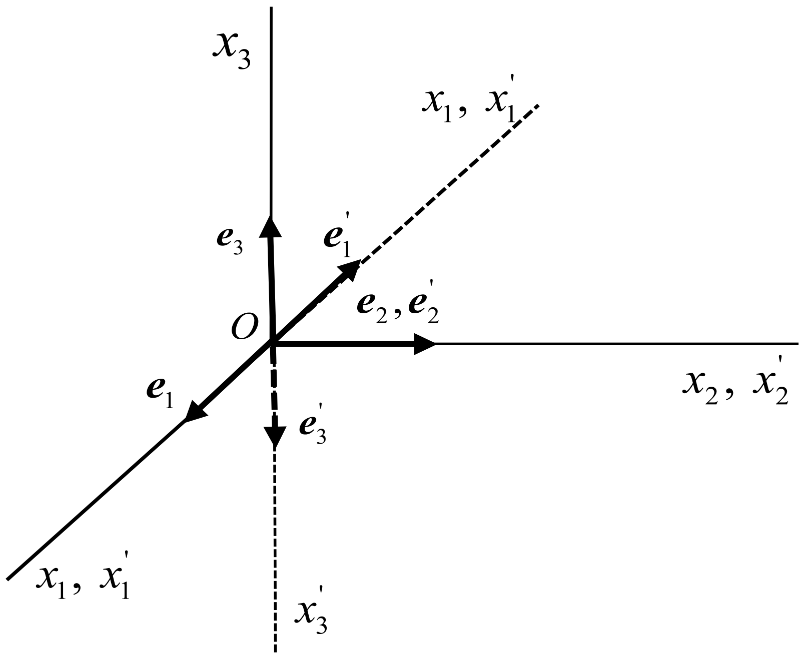
Transforming the matrix of the elastic constants and imposing the symmetry requirement the number of the elastic constants reduces to *thirteen*.

Typical examples are natural materials like**,** kaolin (a clay material) and muscovite (or mica).

Mechanics of Solids: Anisotropy

BASIC CASES OF ELASTIC SYMMETRY

2: symmetry with respect to two orthogonal planes:
material is defined as orthotropic



$$\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{13}\\ \sigma_{23}\end{pmatrix} = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0\\ C_{1122} & C_{2222} & C_{2233} & 0 & 0 & 0\\ C_{1133} & C_{2233} & C_{3333} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{1212} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{1313} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{2323}\end{pmatrix}\begin{pmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{12}\\ 2\varepsilon_{13}\\ 2\varepsilon_{23}\end{pmatrix}$$

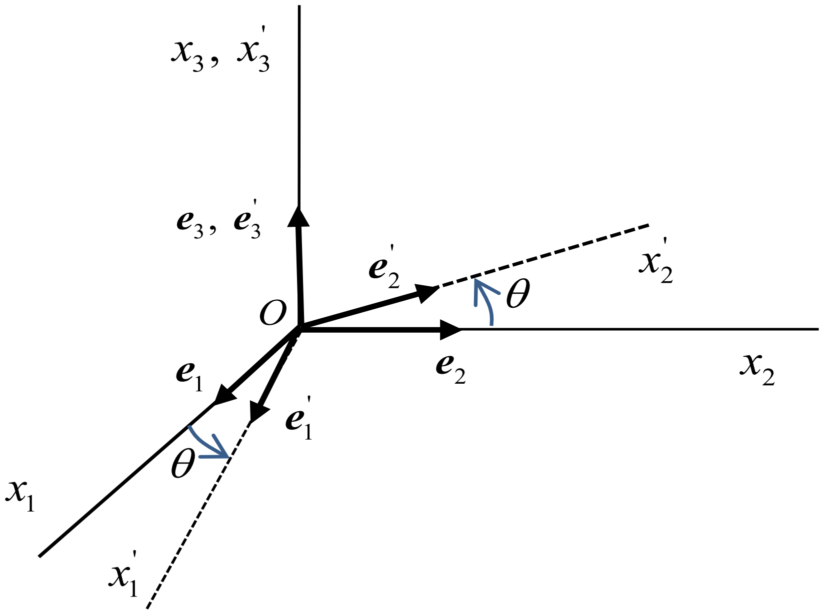
Transforming the matrix of the elastic constants and imposing the symmetry requirement the number of the elastic constants reduces to *nine*.

Here we find materials like wood, layered composite materials, rolled metals.

Mechanics of Solids: Anisotropy

BASIC CASES OF ELASTIC SYMMETRY

3: symmetry with respect to one axis: *material is transversely isotropic*



$$\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{13}\\ \sigma_{23}\end{pmatrix}=\begin{pmatrix}C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0\\ C_{1122} & C_{1111} & C_{1133} & 0 & 0 & 0\\ C_{1133} & C_{1133} & C_{3333} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{2}(C_{1111}-C_{1122}) & 0 & 0\\ 0 & 0 & 0 & 0 & C_{1313} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{1313}\end{pmatrix}\begin{pmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{12}\\ 2\varepsilon_{13}\\ 2\varepsilon_{23}\end{pmatrix}$$

Transforming the matrix of the elastic constants and imposing the symmetry requirement the number of the elastic constants reduces to *five*.

Here we find soil materials (deposited layers..)

Mechanics of Solids: Anisotropy

Basic Cases of Elastic Symmetry

4: symmetry with respect to all axes (independent of orientation): *material is isotropic*

Here we find only two independent constants λ, μ :

$$C_{1111} = \lambda + 2\mu \;\; ; \;\; C_{1122} = \lambda \;\; ; \;\; C_{1212} = (C_{1111} - C_{1212}) = 2\mu$$

λ, μ are called Lamé constants and related to young modulus E and Poisson's ratio v :

$$\lambda = Ev/(1+v)(1-2v) \qquad \mu = E/2(1+v)$$

Here we find several materials metals, ceramics
Polymers, particulate composites,……

Mechanics of Solids: Anisotropy

ISOTROPIC MATERIAL

Hook's Law: linear isotropic solid

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{pmatrix} = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{pmatrix}$$

in index form

$$\sigma_{kl} = \lambda \varepsilon_{pp} \delta_{kl} + 2\mu \varepsilon_{kl}$$

its inverse

$$\varepsilon_{ij} = -\frac{\lambda \delta_{ij}}{2\mu(3\lambda+2\mu)} \sigma_{kk} + \frac{1}{2\mu} \sigma_{ij}$$

two elastic constants.

$$\sigma_{kl} = C_{klmn} \varepsilon_{mn} = \left[\lambda \delta_{kl} \delta_{mn} + \mu (\delta_{km} \delta_{ln} + \delta_{kn} \delta_{lm}) \right] \varepsilon_{mn}$$
$$= \lambda \varepsilon_{mm} \delta_{kl} + 2\mu \varepsilon_{kl} \ .$$

$$C_{klmn} = \lambda \delta_{kl} \delta_{mn} + \mu (\delta_{km} \delta_{ln} + \delta_{kn} \delta_{lm})$$

Isotropic forth order tensor

ISOTROPIC MATERIAL

Strain energy density: linear isotropic solid

$$W(\varepsilon_{ij}) = \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{kk} + \mu\varepsilon_{ij}\varepsilon_{ij}$$

It is a stress potential
for linear elastic material

$$\sigma_{pq} = \frac{\partial W(\varepsilon_{ij})}{\partial\varepsilon_{pq}} = \frac{1}{2}\lambda\left(\frac{\partial\varepsilon_{ii}}{\partial\varepsilon_{pq}}\varepsilon_{kk} + \varepsilon_{ii}\frac{\partial\varepsilon_{kk}}{\partial\varepsilon_{pq}}\right) + 2\mu\varepsilon_{ij}\frac{\partial\varepsilon_{ij}}{\partial\varepsilon_{pq}}$$

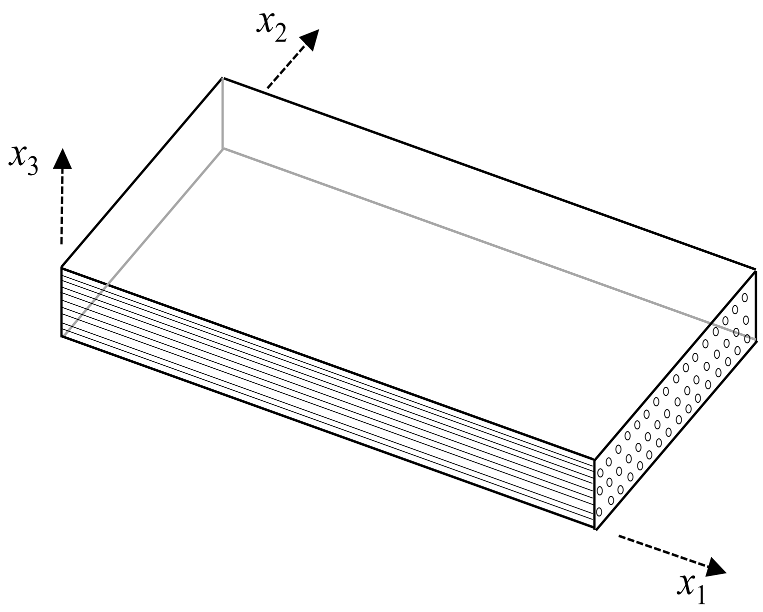
$$\sigma_{pq} = \frac{1}{2}\lambda\left(\delta_{ip}\delta_{iq}\varepsilon_{kk} + \varepsilon_{ii}\delta_{kp}\delta_{kq}\right) + 2\mu\varepsilon_{ij}\delta_{ip}\delta_{jq}$$

$$= \frac{1}{2}\lambda\left(2\delta_{pq}\varepsilon_{kk}\right) + 2\mu\varepsilon_{pq} = \lambda\delta_{pq}\varepsilon_{kk} + 2\mu\varepsilon_{pq}$$

Mechanics of Solids: Anisotropy-application to composites

Basic Cases of Elastic Symmetry

Typical lamina: *orthotropic symmetry*



the number of the elastic constants is *nine*.

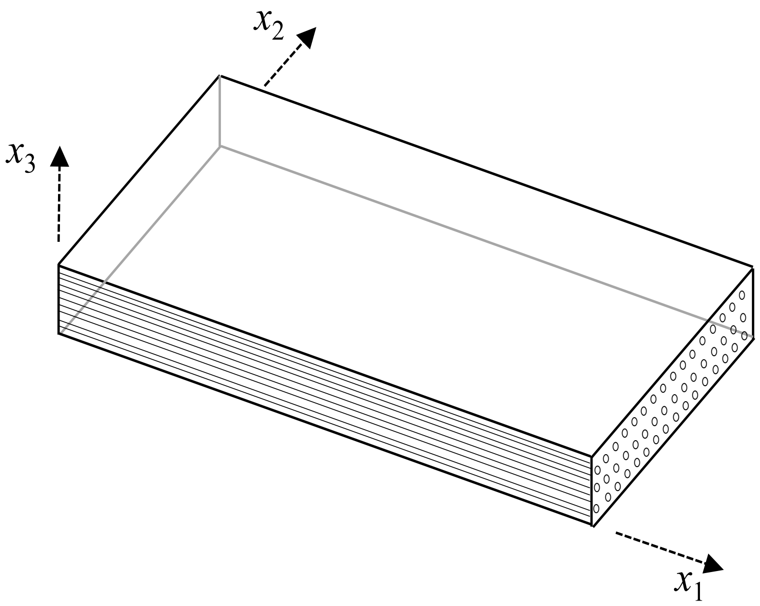
$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{pmatrix} = \begin{pmatrix} S_{1111} & S_{1122} & S_{1133} & 0 & 0 & 0 \\ S_{1122} & S_{2222} & S_{2233} & 0 & 0 & 0 \\ S_{1133} & S_{2233} & S_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{1212} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{2323} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ 2\sigma_{12} \\ 2\sigma_{13} \\ 2\sigma_{23} \end{pmatrix}$$

remarks:

1. There is no coupling between normal stresses and shear strains.
2. There is no coupling between shear stresses and normal strains.
3. There is no coupling between a shear stress acting on one plane
4. and a shear stress on a different plane.

Mechanics of Solids: Anisotropy- application to composites

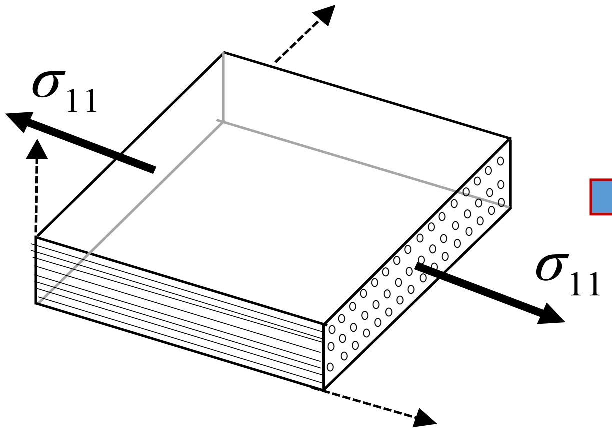
To determine the elastic constants in the stress strain relation we perform 6 elementary experiments



$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{pmatrix} = \begin{pmatrix} S_{1111} & S_{1122} & S_{1133} & 0 & 0 & 0 \\ S_{1122} & S_{2222} & S_{2233} & 0 & 0 & 0 \\ S_{1133} & S_{2233} & S_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{1212} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{2323} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ 2\sigma_{12} \\ 2\sigma_{13} \\ 2\sigma_{23} \end{pmatrix}$$

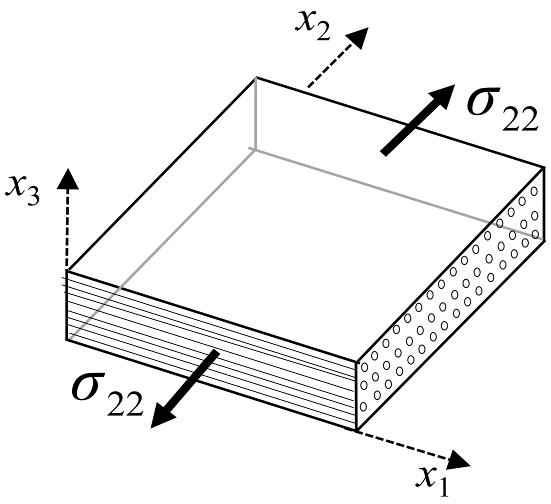
We apply on a small specimen of composite material, single tractions and shear stresses along the three directions one at a time.

Mechanics of Solids: Anisotropy-application to composites



$$\varepsilon_{11} = S_{1111}\sigma_{11} = \frac{1}{E_1}\sigma_{11} \quad ; \quad \varepsilon_{22} = S_{1122}\sigma_{11} = -\frac{\nu_{12}}{E_1}\sigma_{11}$$

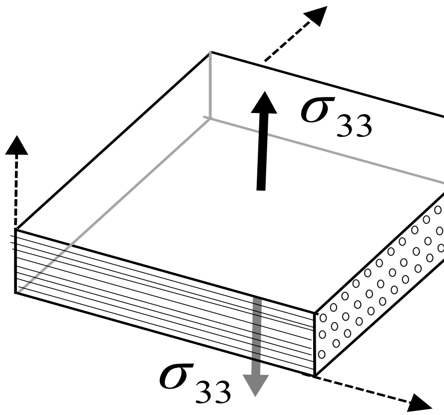
$$\varepsilon_{33} = S_{1133}\sigma_{11} = -\frac{\nu_{13}}{E_1}\sigma_{11} \quad ; \quad \varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = 0$$



$$\varepsilon_{11} = S_{1122}\sigma_{22} = -\frac{\nu_{21}}{E_2}\sigma_{22} \quad ; \quad \varepsilon_{22} = S_{2222}\sigma_{22} = \frac{1}{E_2}\sigma_{22}$$

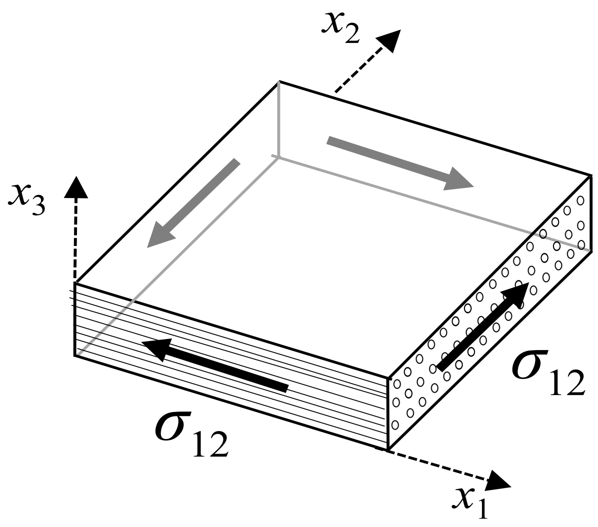
$$\varepsilon_{33} = S_{1133}\sigma_{33} = -\frac{\nu_{23}}{E_2}\sigma_{22} \quad ; \quad \varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = 0$$

Mechanics of Solids: Anisotropy-application to composites



$$\varepsilon_{11} = S_{1133}\sigma_{33} = -\frac{\nu_{31}}{E_3}\sigma_{33} \quad ; \quad \varepsilon_{22} = S_{2233}\sigma_{33} = -\frac{\nu_{32}}{E_3}\sigma_{33}$$

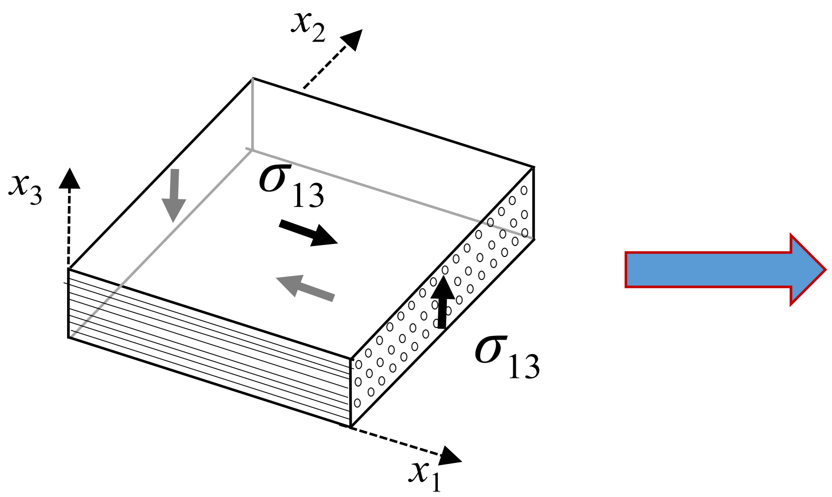
$$\varepsilon_{33} = S_{1133}\sigma_{33} = \frac{1}{E_3}\sigma_{33} \quad ; \quad \varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = 0$$



$$\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = 0$$

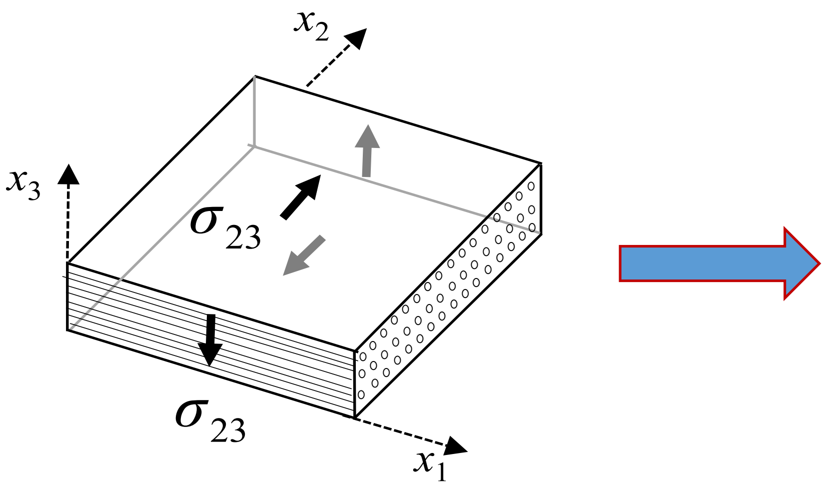
$$\varepsilon_{12} = \frac{1}{2G_{12}}\sigma_{12} \quad ; \quad \varepsilon_{13} = \varepsilon_{23} = 0$$

Mechanics of Solids: Anisotropy-application to composites



$$\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = 0$$

$$\varepsilon_{13} = \frac{1}{2G_{13}}\sigma_{13} \quad ; \quad \varepsilon_{12} = \varepsilon_{23} = 0$$

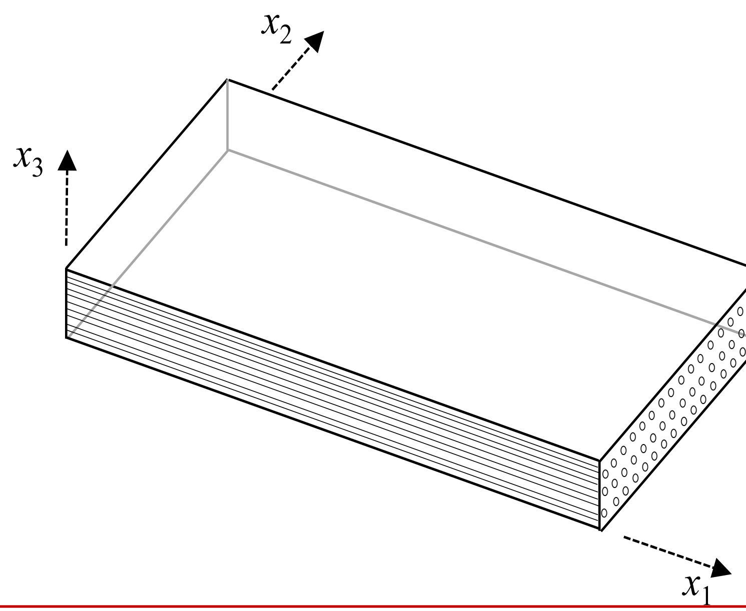


$$\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = 0$$

$$\varepsilon_{23} = \frac{1}{2G_{23}}\sigma_{23} \quad ; \quad \varepsilon_{12} = \varepsilon_{13} = 0$$

Mechanics of Solids: Anisotropy-application to composites

orthotropic symmetry



In terms of engineering constants

$$
\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{pmatrix} =
\begin{pmatrix}
\frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\
-\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\
-\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{4G_{12}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{4G_{13}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{4G_{23}}
\end{pmatrix}
\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ 2\sigma_{12} \\ 2\sigma_{13} \\ 2\sigma_{23} \end{pmatrix}
$$

$E_1, E_2, E_3, \quad G_{12}, G_{13}, G_{23},$

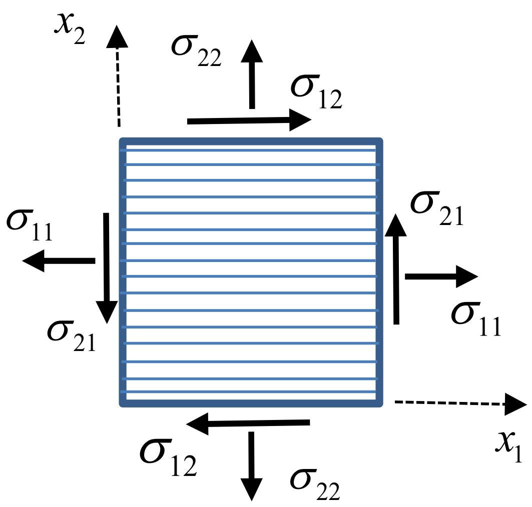
$\nu_{21}, \nu_{12}, \nu_{13}, \nu_{31}, \nu_{23}, \nu_{32}$

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1} \quad ; \quad \frac{\nu_{13}}{E_1} = \frac{\nu_{31}}{E_3} \quad ; \quad \frac{\nu_{23}}{E_2} = \frac{\nu_{32}}{E_3}$$

NINE independent elastic constants.

Mechanics of Solids: Anisotropy-application to composites

Composite plates: Conditions of plane stress



$$\sigma_{11},\ \sigma_{22},\ \sigma_{12}=\sigma_{21}\neq 0$$

$$\sigma_{13},\ \sigma_{23},\ \sigma_{33}=0$$

$$\begin{pmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{12}\end{pmatrix}=\begin{pmatrix}S_{1111} & S_{1122} & 0\\ S_{1122} & S_{2222} & 0\\ 0 & 0 & S_{1212}\end{pmatrix}\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ 2\sigma_{12}\end{pmatrix}$$

$$\begin{pmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{12}\end{pmatrix}=\begin{pmatrix}\dfrac{1}{E_1} & -\dfrac{v_{21}}{E_2} & 0\\ -\dfrac{v_{12}}{E_1} & \dfrac{1}{E_2} & 0\\ 0 & 0 & \dfrac{1}{4G_{12}}\end{pmatrix}\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ 2\sigma_{12}\end{pmatrix}$$

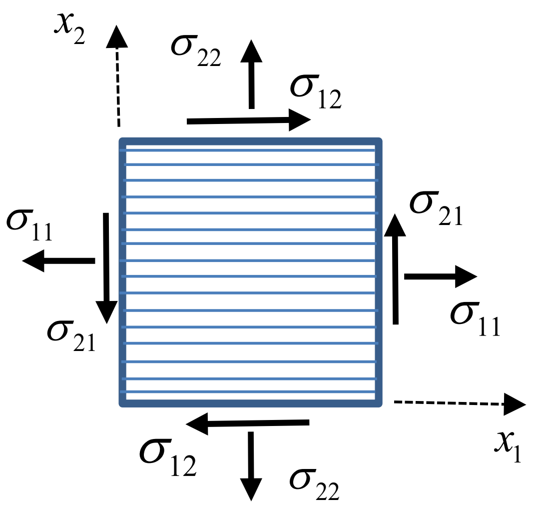
Four independent elastic constants

$$E_1, E_2, G_{12}, v_{21}, v_{12}$$

$$v_{21}/E_2 = v_{12}/E_1$$

Mechanics of Solids: Anisotropy-application to composites

Composite plates: simplifications



$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{pmatrix} \rightarrow \begin{pmatrix} \varepsilon_{1} \\ \varepsilon_{2} \\ \gamma_{12} \end{pmatrix}$$

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} \rightarrow \begin{pmatrix} \sigma_{1} \\ \sigma_{2} \\ \sigma_{12} \end{pmatrix}$$

$$\begin{pmatrix} \varepsilon_{1} \\ \varepsilon_{2} \\ \gamma_{12} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{pmatrix} \begin{pmatrix} \sigma_{1} \\ \sigma_{2} \\ \sigma_{12} \end{pmatrix}$$

With

$$S_{11} = \frac{1}{E_1}; \; S_{22} = \frac{1}{E_2};$$

$$S_{12} = S_{21} = -\frac{\nu_{21}}{E_2} = -\frac{\nu_{12}}{E_1};$$

$$S_{66} = \frac{1}{G_{12}}$$

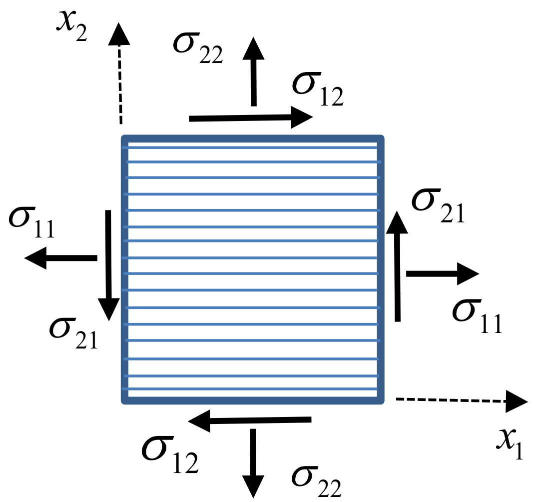
Note here the shear component: $\gamma_{12} = S_{66}\sigma_{12}$

It is so designated to 'comply' with some books in composites

The stress-strain relations are described with reference to the principal material system.

Mechanics of Solids: Anisotropy-application to composites

Composite plates: simplifications in terms of stiffness



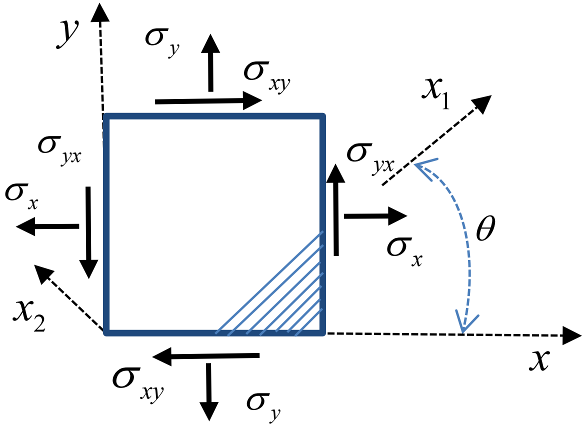
$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{pmatrix} \Rightarrow \begin{pmatrix} \sigma_{1} \\ \sigma_{2} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{1} \\ \varepsilon_{2} \\ \gamma_{12} \end{pmatrix}$$

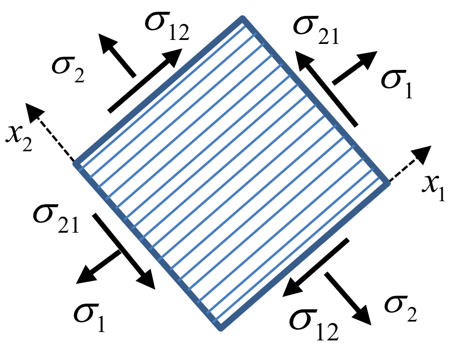
$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}} \quad ; \quad Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}$$

$$Q_{12} = Q_{21} = \frac{\nu_{21}E_1}{1 - \nu_{12}\nu_{21}} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}} \quad ; \quad Q_{66} = G_{12}$$

Mechanics of Solids: Anisotropy-application to composites

Composite plates: changes of coordinate system





The stress-strain relations are described with reference to the principal material system.

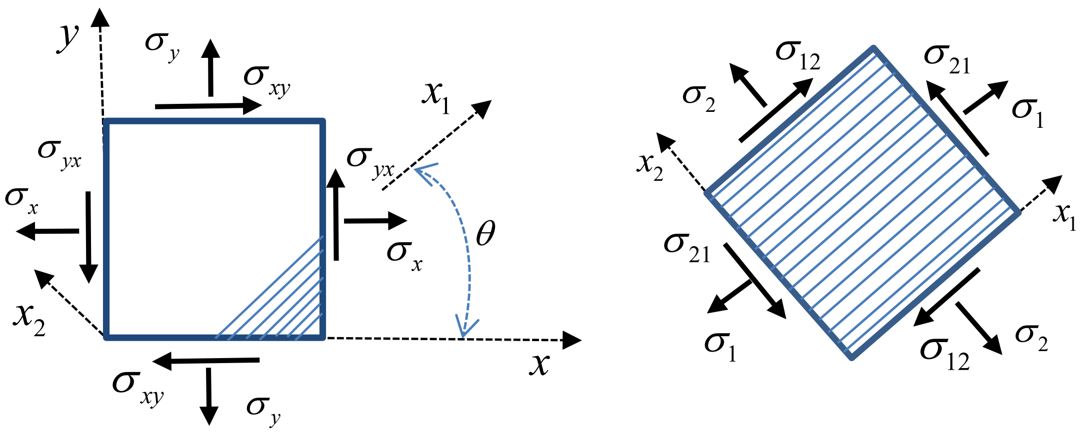
$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{pmatrix}$$

The stress-strain relations are described with reference to a rotated system.

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} S_{xx} & S_{xy} & S_{xs} \\ S_{yx} & S_{yy} & S_{ys} \\ S_{sx} & S_{sy} & S_{ss} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix}$$

Mechanics of Solids: Anisotropy-application to composites

Composite plates: changes of coordinate system



For the stresses:

$$\sigma_1 = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\sigma_{xy}\cos\theta\sin\theta$$

$$\sigma_2 = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\sigma_{xy}\cos\theta\sin\theta$$

$$\sigma_{12} = \sigma_{21} = -(\sigma_x - \sigma_y)\cos\theta\sin\theta + \sigma_{xy}(\cos^2\theta - \sin^2\theta).$$

For the strains:

$$\varepsilon_1 = \varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta + \varepsilon_{xy}\cos\theta\sin\theta$$

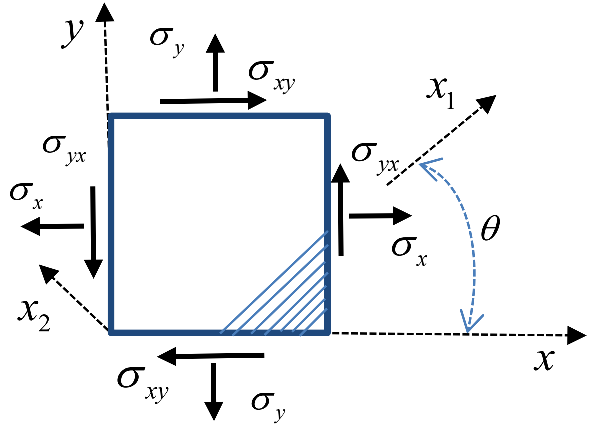
$$\varepsilon_2 = \varepsilon_x \sin^2\theta + \sigma_y \cos^2\theta - \varepsilon_{xy}\cos\theta\sin\theta$$

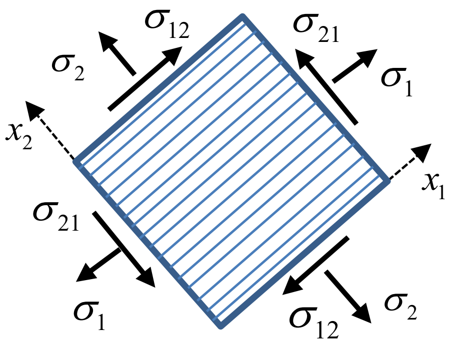
$$\varepsilon_{12} = -(\varepsilon_x - \varepsilon_y)\cos\theta\sin\theta + \varepsilon_{xy}(\cos^2\theta - \sin^2\theta).$$

Mechanics of Solids: Anisotropy-application to composites

Composite plates: changes of coordinate system

Stiffness and compliance matrices





with $c = \cos\theta; \quad s = \sin\theta$

For the stiffness matrix

$$Q_{xx} = Q_{11}c^4 + Q_{22}s^4 + (2Q_{12} + 4Q_{66})c^2s^2$$

$$Q_{yy} = Q_{11}s^4 + Q_{22}c^4 + (2Q_{12} + 4Q_{66})c^2s^2$$

$$Q_{ss} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})c^2s^2 + Q_{66}(c^4 + s^4)$$

$$Q_{xy} = (Q_{11} + Q_{22} - 4Q_{66})c^2s^2 + Q_{12}(c^4 + s^4)$$

$$Q_{xs} = (Q_{11} - Q_{12} - 2Q_{66})c^3s - (Q_{22} - Q_{12} - 2Q_{66})cs^3$$

$$Q_{ys} = (Q_{11} - Q_{12} - 2Q_{66})cs^3 - (Q_{22} - Q_{12} - 2Q_{66})c^3s.$$

For the compliance matrix:

$$S_{xx} = S_{11}c^4 + S_{22}s^4 + (2S_{12} + S_{66})c^2s^2$$

$$S_{yy} = S_{11}s^4 + S_{22}c^4 + (2S_{12} + S_{66})c^2s^2$$

$$S_{ss} = (4S_{11} + 4S_{22} - 8S_{12} - 2S_{66})c^2s^2 + S_{66}(c^4 + s^4)$$

$$S_{xy} = (S_{11} + S_{22} - S_{66})c^2s^2 + S_{12}(c^4 + s^4)$$

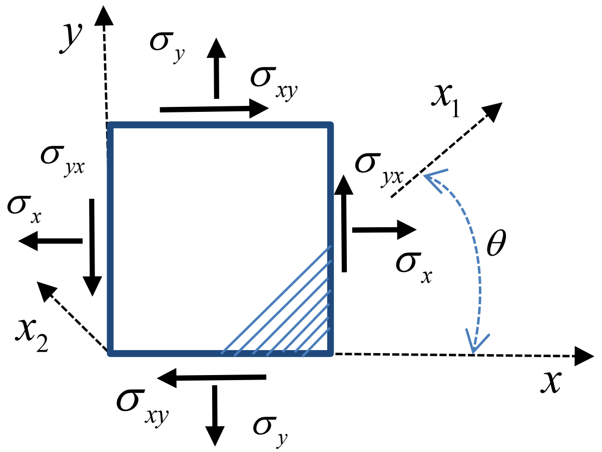
$$S_{xs} = (2S_{11} - 2S_{12} - S_{66})c^3s - (2S_{22} - 2S_{12} - S_{66})cs^3$$

$$S_{ys} = (2S_{11} - 2S_{12} - S_{66})cs^3 - (2S_{22} - 2S_{12} - S_{66})c^3s.$$

Mechanics of Solids: Anisotropy-application to composites

Composite plates: changes of coordinate system

engineering constants (i.e. moduli and Poisson's ratios)
with reference to a coordinate system rotated at an angle 'theta' with respect to
the principal material coordinates. We can load the plate with single loads (normal and shear)
to obtain the stress strain relations (similar to the 3D case shown earlier)



$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & \frac{\eta_{sx}}{G_{xy}} \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & \frac{\eta_{sy}}{G_{xy}} \\ \frac{\eta_{xs}}{E_x} & \frac{\eta_{ys}}{E_y} & \frac{1}{G_{xy}} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix}$$

With the symmetry conditions

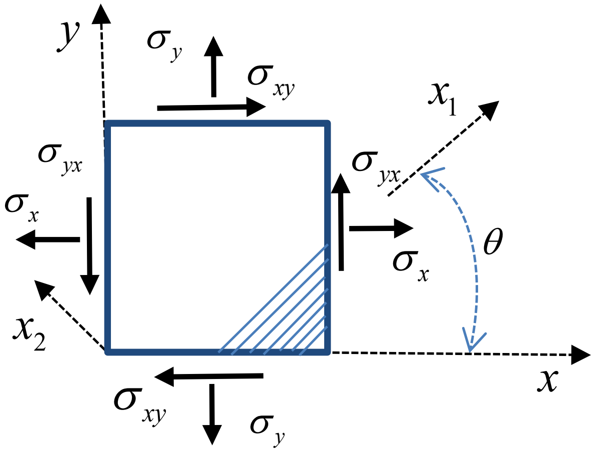
$$\frac{\nu_{xy}}{E_x} = \frac{\nu_{yx}}{E_y}$$

$$\frac{\eta_{xs}}{E_x} = \frac{\eta_{sx}}{G_{xy}}$$

$$\frac{\eta_{ys}}{E_y} = \frac{\eta_{sy}}{G_{xy}}$$

Note that the matrix is full!

Mechanics of Solids: Anisotropy-application to composites



$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & \frac{\eta_{sx}}{G_{xy}} \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & \frac{\eta_{sy}}{G_{xy}} \\ \frac{\eta_{xs}}{E_x} & \frac{\eta_{ys}}{E_y} & \frac{1}{G_{xy}} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix}$$

Compare with

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} S_{xx} & S_{xy} & S_{xs} \\ S_{yx} & S_{yy} & S_{ys} \\ S_{sx} & S_{sy} & S_{ss} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix}$$

$$S_{xx} = \frac{1}{E_x}; \quad S_{yy} = \frac{1}{E_y}; \quad S_{ss} = \frac{1}{G_{xy}}; \quad S_{xy} = S_{yx} = -\frac{\nu_{xy}}{E_x} = -\frac{\nu_{yx}}{E_y}$$

$$S_{xs} = S_{sx} = \frac{\eta_{xs}}{E_x} = \frac{\eta_{sx}}{G_{xy}}; \qquad S_{ys} = S_{sy} = \frac{\eta_{ys}}{E_y} = \frac{\eta_{sy}}{G_{xy}}$$

Mechanics of Solids: Anisotropy-application to composites

We can relate the elastic constants (moduli and Poisson's ratios) in the rorated system of coordinates to The principal system of coordinates. We need to combine the following relations:

$$S_{11}=\frac{1}{E_1};\; S_{22}=\frac{1}{E_2};\; S_{12}=S_{21}=-\frac{\nu_{21}}{E_2}=-\frac{\nu_{12}}{E_1};\; S_{66}=\frac{1}{G_{12}}$$

$$S_{xx}=S_{11}c^4+S_{22}s^4+\left(2S_{12}+S_{66}\right)c^2s^2$$
$$S_{yy}=S_{11}s^4+S_{22}c^4+\left(2S_{12}+S_{66}\right)c^2s^2$$
$$S_{ss}=\left(4S_{11}+4S_{22}-8S_{12}-2S_{66}\right)c^2s^2+S_{66}\left(c^4+s^4\right)$$
$$S_{xy}=\left(S_{11}+S_{22}-S_{66}\right)c^2s^2+S_{12}\left(c^4+s^4\right)$$
$$S_{xs}=\left(2S_{11}-2S_{12}-S_{66}\right)c^3s-\left(2S_{22}-2S_{12}-S_{66}\right)cs^3$$
$$S_{ys}=\left(2S_{11}-2S_{12}-S_{66}\right)cs^3-\left(2S_{22}-2S_{12}-S_{66}\right)c^3s.$$

$$S_{xx}=\frac{1}{E_x};\quad S_{yy}=\frac{1}{E_y};\quad S_{ss}=\frac{1}{G_{xy}};\quad S_{xy}=S_{yx}=-\frac{\nu_{xy}}{E_x}=-\frac{\nu_{yx}}{E_y}$$

$$S_{xs}=S_{sx}=\frac{\eta_{xs}}{E_x}=\frac{\eta_{sx}}{G_{xy}};\qquad S_{ys}=S_{sy}=\frac{\eta_{ys}}{E_y}=\frac{\eta_{sy}}{G_{xy}}$$

$$\Rightarrow \frac{1}{E_x}=\frac{c^2}{E_1}\left(c^2-\nu_{12}s^2\right)+\frac{s^2}{E_2}\left(s^2-\nu_{21}c^2\right)+\frac{1}{G_{12}}c^2s^2.$$

Mechanics of Solids: Anisotropy-application to composites

In a similar manner we can obtain we have for the other elastic constants:

$$\frac{1}{E_x} = \frac{c^2}{E_1}\left(c^2 - v_{12}s^2\right) + \frac{s^2}{E_2}\left(s^2 - v_{21}c^2\right) + \frac{1}{G_{12}}c^2 s^2$$

$$\frac{1}{E_y} = \frac{s^2}{E_1}\left(s^2 - v_{12}c^2\right) + \frac{c^2}{E_2}\left(c^2 - v_{21}s^2\right) + \frac{1}{G_{12}}c^2 s^2$$

$$\frac{1}{G_{xy}} = \frac{4s^2c^2}{E_1}\left(1 + v_{12}\right) + \frac{4s^2c^2}{E_2}\left(1 + v_{21}\right) + \frac{(c^2 - s^2)^2}{G_{12}}$$

$$\frac{v_{xy}}{E_x} = \frac{v_{yx}}{E_y} = \frac{c^2}{E_1}\left(v_{12}c^2 - s^2\right) + \frac{s^2}{E_2}\left(v_{21}s^2 - c^2\right) + \frac{1}{G_{12}}c^2 s^2$$

$$\frac{\eta_{xs}}{E_x} = \frac{\eta_{sx}}{G_{xy}} = \frac{2c^3 s}{E_1}\left(1 + v_{12}\right) - \frac{2cs^3}{E_2}\left(1 + v_{21}\right) - \frac{cs}{G_{12}}\left(c^2 - s^2\right)$$

$$\frac{\eta_{ys}}{E_y} = \frac{\eta_{sy}}{G_{xy}} = \frac{2cs^3}{E_1}\left(1 + v_{12}\right) - \frac{2c^3 s}{E_2}\left(1 + v_{21}\right) + \frac{cs}{G_{12}}\left(c^2 - s^2\right)$$